State-space Model

CEH3K3

Randy E. Saputra, ST. MT.

State-space Model

- Instead of Mathematical Model, dynamic system can be represented with State-space Model

- It consists of:
  - ✓ State Variable, which is making up the smallest set of variables that determine the state of the dynamic system
  - ✓ State-space Equation, concerned with three types of variables; input variables, output variables, and state variables

General Form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

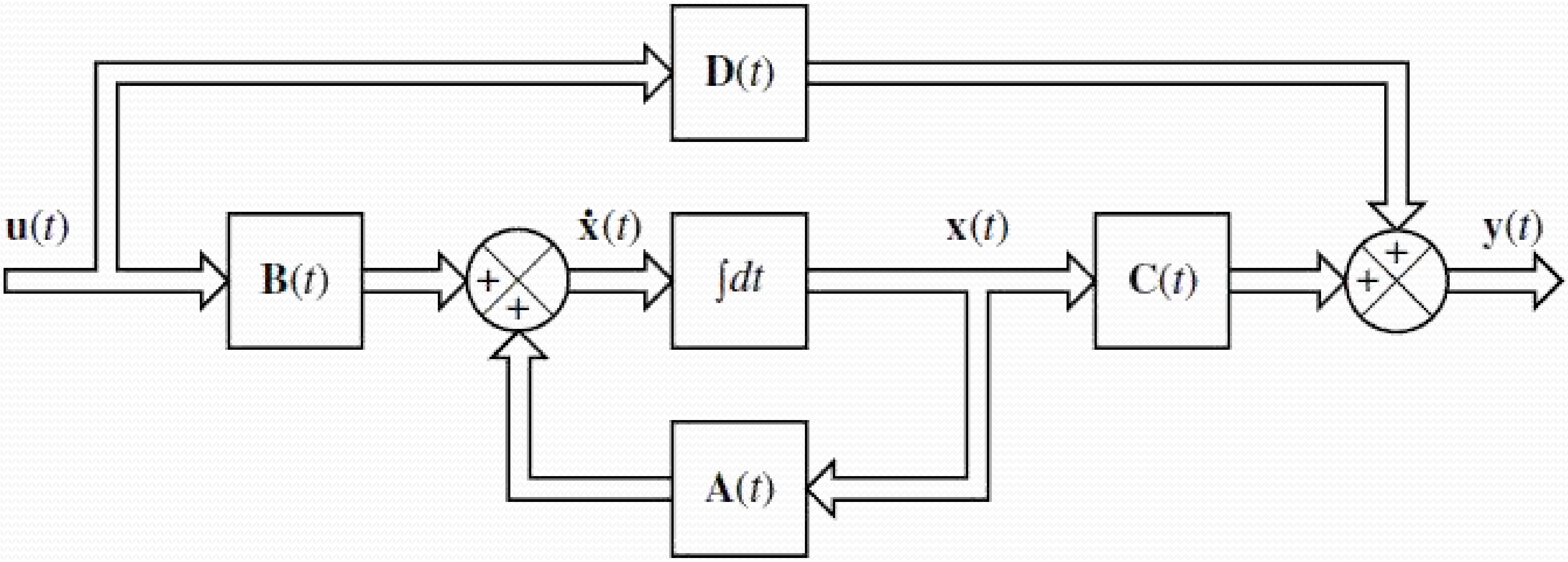
A is state matrix

B is input matrix

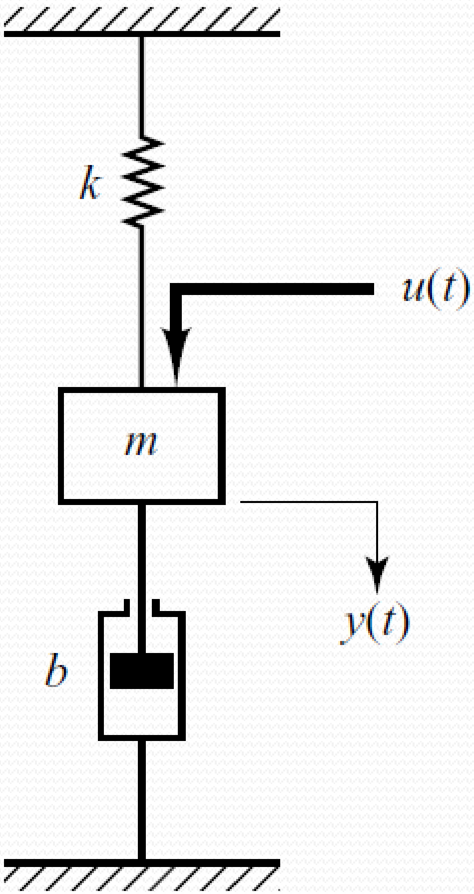
C is output matrix

D is direct transmission matrix

General Form



Example



Find the state-space equation of the mechanical system shown in the diagram while the system equation is

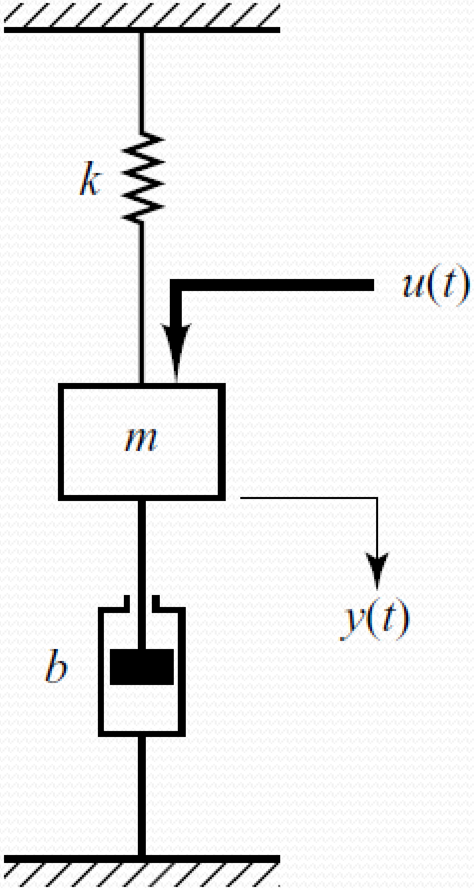
$$m\ddot{y} + b\dot{y} + ky = u$$

Solution:

This system is *second order*, so we need two integrator. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

Example



Then we obtain

$$\dot{x}_1 = x_2$$

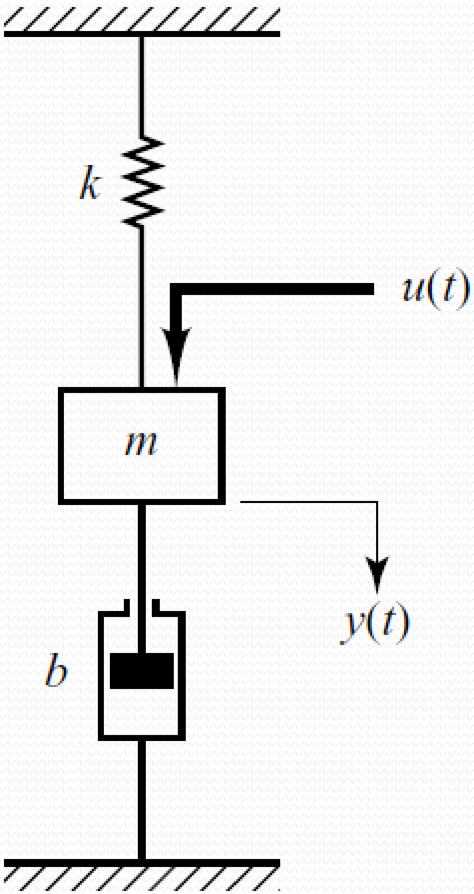
$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

or

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

Example



The output equation is

$$y = x_1$$

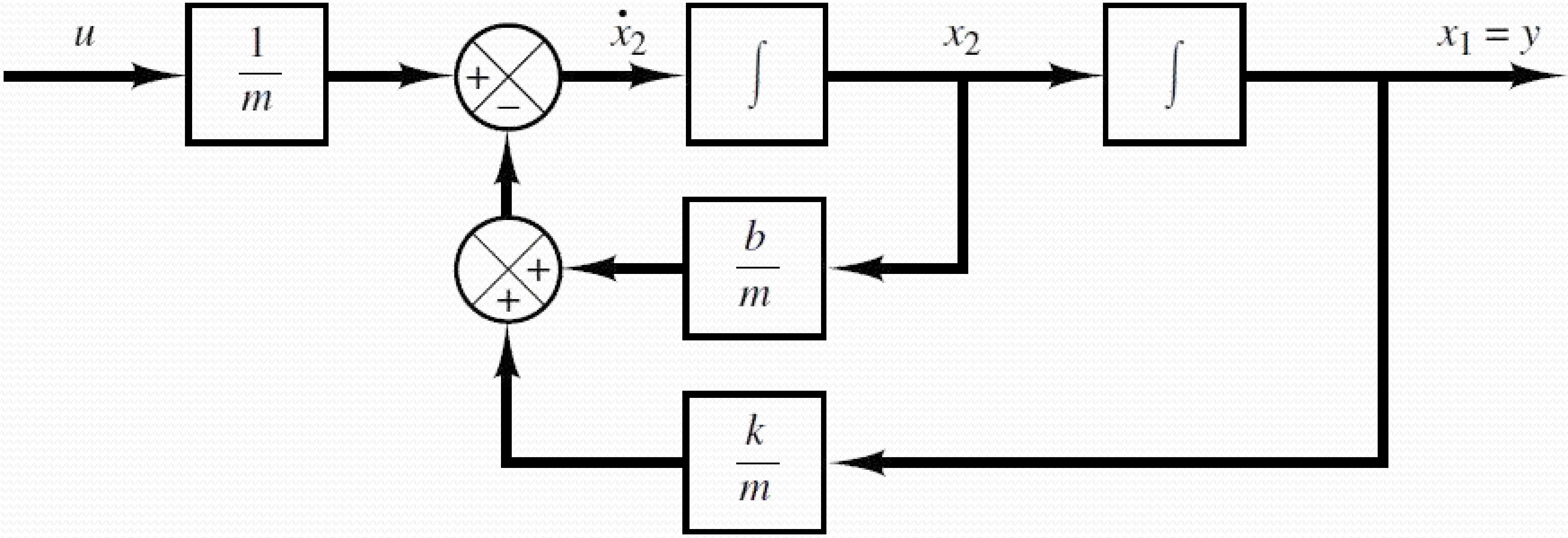
So the matrix form can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example

The block diagram of this system is



Correlation between Transfer Function and State-space Equation

As we know, the general form of Transfer Function is

$$\frac{Y(s)}{U(s)} = G(s)$$

and the State-space Equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

Correlation between Transfer Function and State-space Equation

The Laplace Transform of State-space Equation is

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

and since $x(0) = 0$, then

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

Correlation between Transfer Function and State-space Equation

By premultiplying $(s\mathbf{I} - \mathbf{A})^{-1}$ to both side, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

then we substitute into output equation, we get

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s)$$

to, the Transfer Function $G(s)$ is

$$\boxed{G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D}$$

Example

Find the Transfer Function of the State-space equation we obtain before.

Answer :

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} + 0$$

$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ \dfrac{k}{m} & s + \dfrac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}$$

Example

Note that

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

Thus we have

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{1}{ms^2 + bs + k}$$

Exercise

- Obtain the transfer function $G(s)$ of the system defined by the following state-space equation :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Exercise

Answer :

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$G(s) = \begin{bmatrix} 1 & 2 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 2 \end{bmatrix} \left(\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 2 \end{bmatrix} \left(\frac{1}{s^2 + 6s + 8} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix} \right) \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Exercise

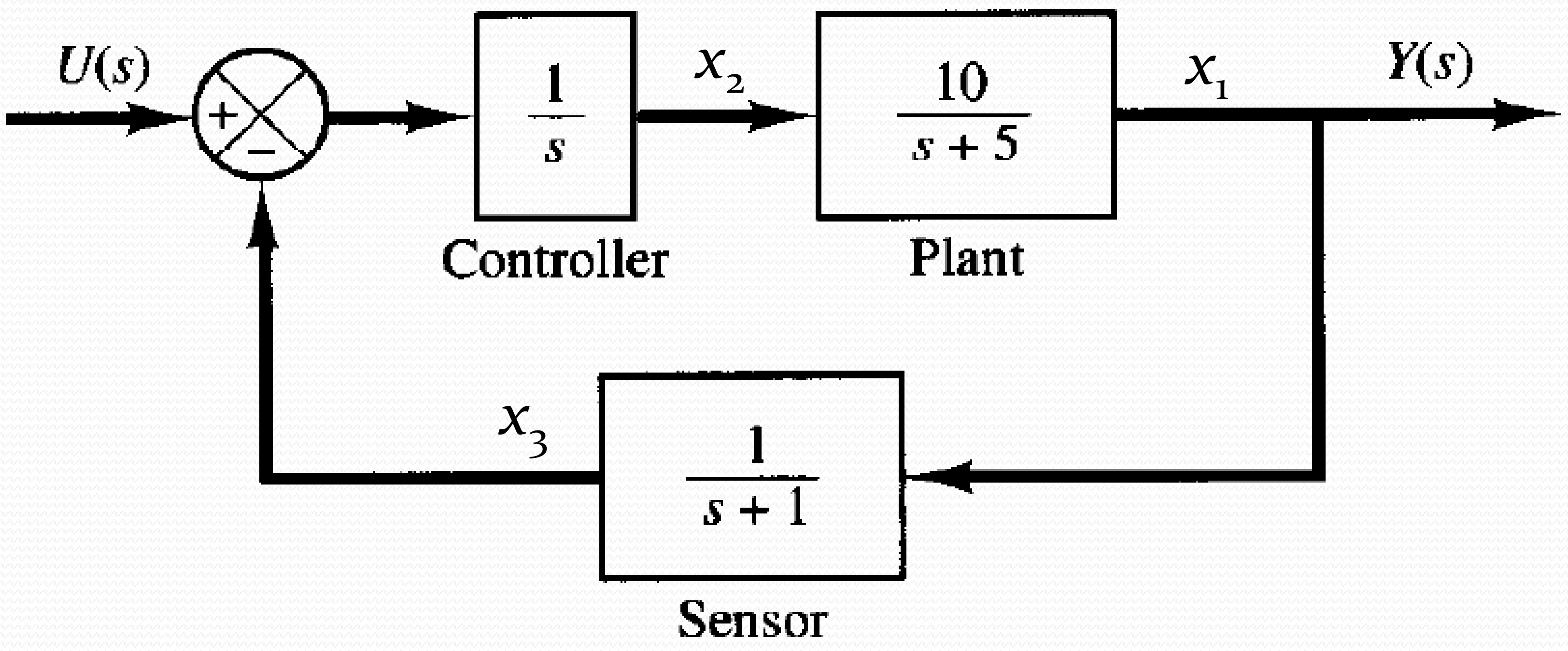
Answer :

$$G(s) = \begin{bmatrix}1 & 2\end{bmatrix} \begin{bmatrix} \dfrac{s+1}{s^2+6s+8} & \dfrac{-1}{s^2+6s+8} \\ \dfrac{3}{s^2+6s+8} & \dfrac{s+5}{s^2+6s+8} \end{bmatrix} \begin{bmatrix}2 \\ 5\end{bmatrix}$$

$$G(s) = \begin{bmatrix} \dfrac{s+7}{s^2+6s+8} & \dfrac{2s+9}{s^2+6s+8} \end{bmatrix} \begin{bmatrix}2 \\ 5\end{bmatrix}$$

$$G(s) = \frac{12s+59}{s^2+6s+8}$$

Obtain State-space Equation from Block Diagram



- Obtain a state-space model of the system!

Obtain State-space Equation from Block Diagram

Write each block equation

$$\frac{X_1(s)}{X_2(s)} = \frac{10}{s+5}$$

$$\frac{X_2(s)}{U(s) - X_3(s)} = \frac{1}{s}$$

$$\frac{X_3(s)}{X_1(s)} = \frac{1}{s+1}$$

$$Y(s) = X_1(s)$$

Obtain State-space Equation from Block Diagram

Simplify these equation, we obtain

$$sX_1(s) = -5X_1(s) + 10X_2(s)$$
$$sX_2(s) = -X_3(s) + U(s)$$
$$sX_3(s) = X_1(s) - X_3(s)$$
$$Y(s) = X_1(s)$$

By Laplace transformation, we can write

$$\dot{x}_1 = -5x_1 + 10x_2$$
$$\dot{x}_2 = -x_3 + u$$
$$\dot{x}_3 = x_1 - x_3$$
$$y = x_1$$

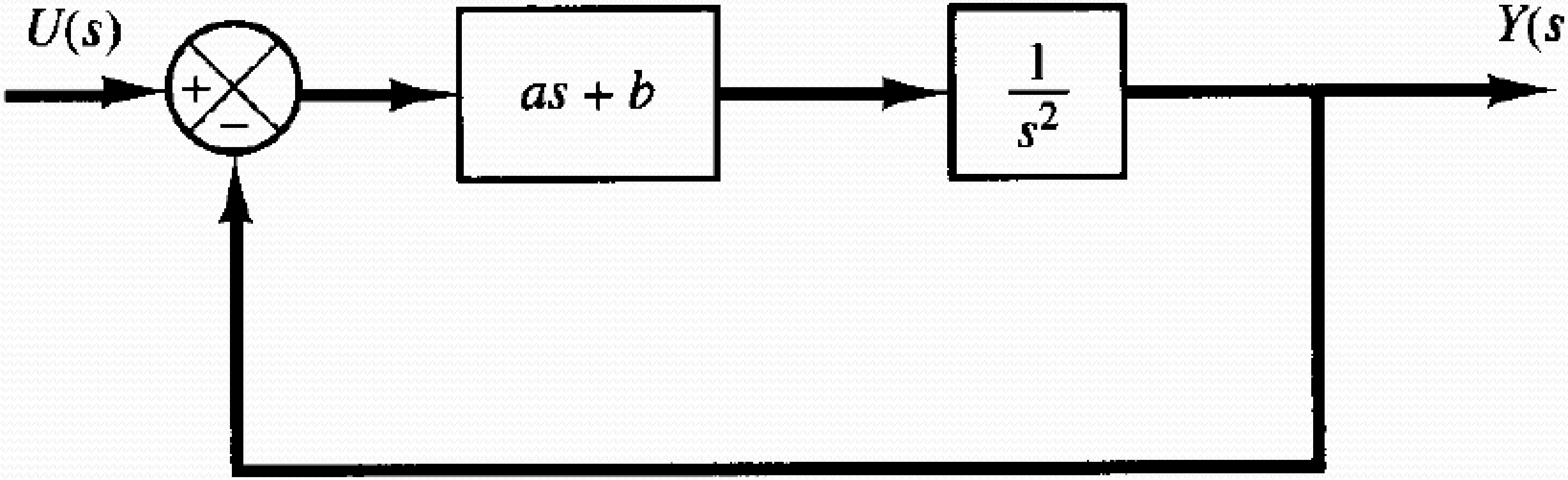
Obtain State-space Equation from Block Diagram

And write it in the standard form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 10 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

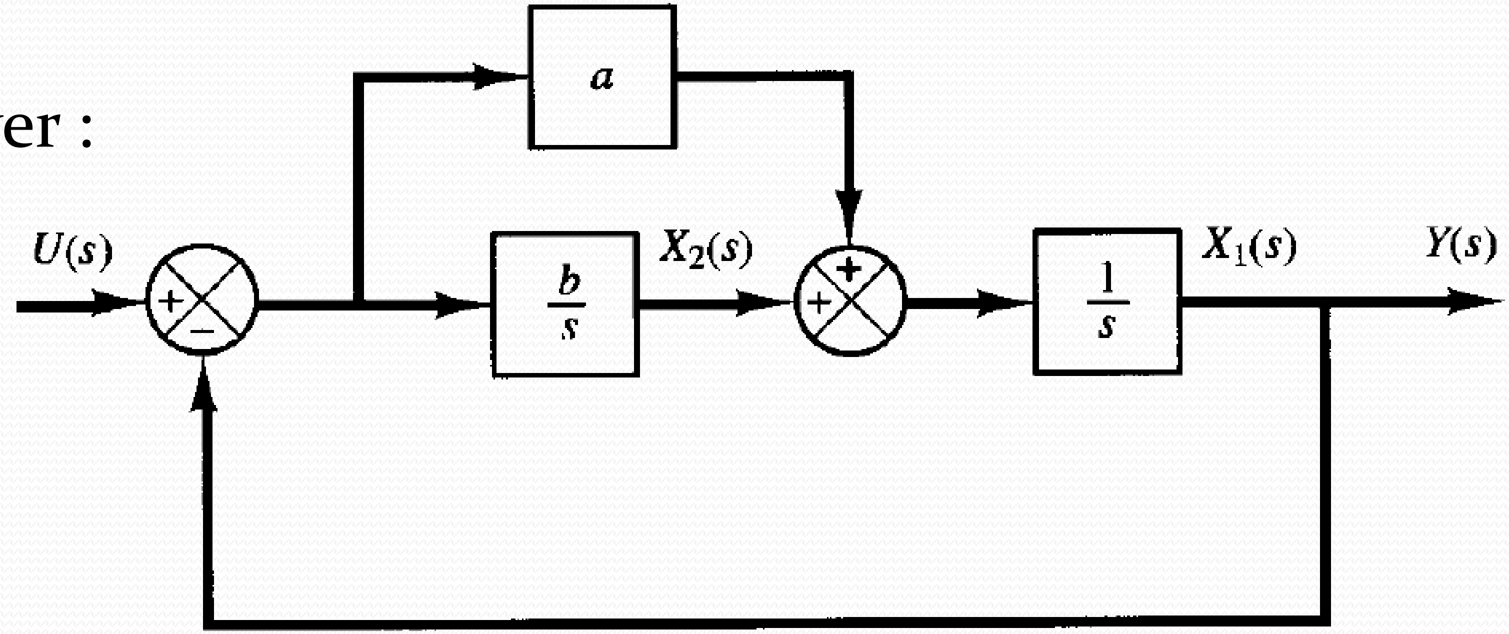
Exercise



- Obtain a state-space model of the system!

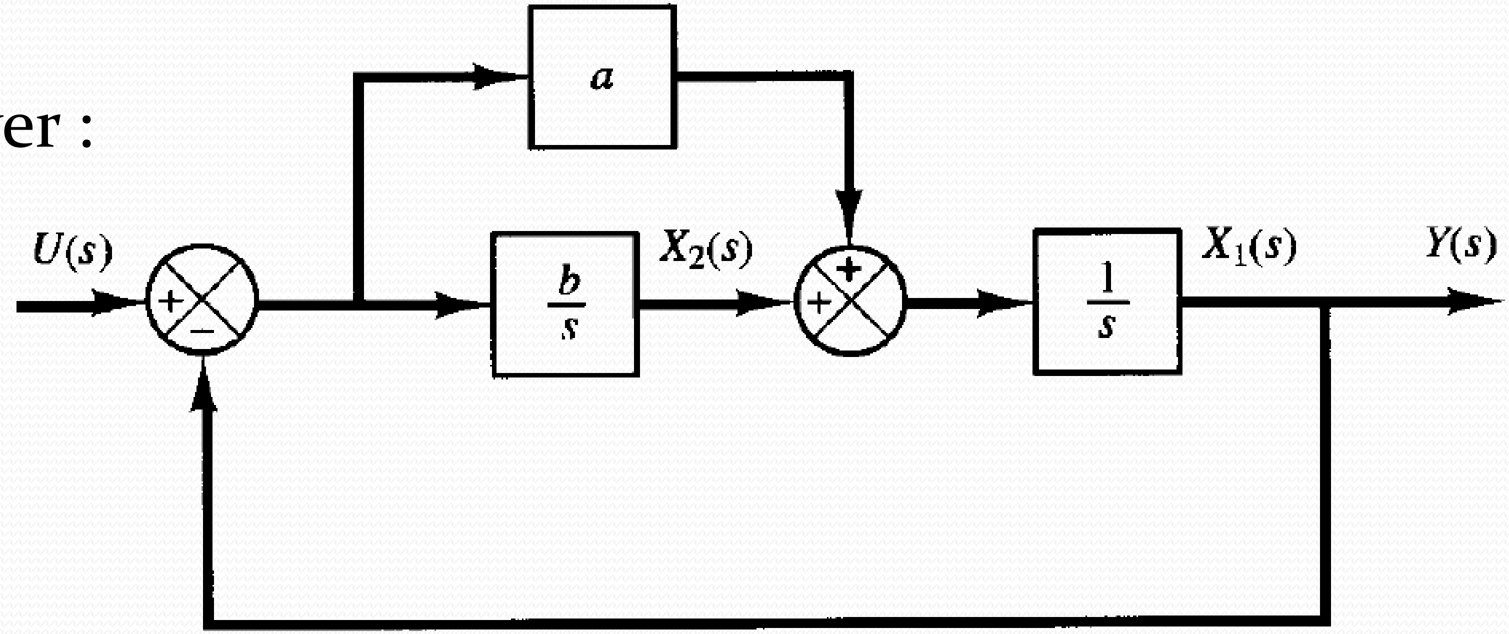
Exercise

- Answer :



Exercise

- Answer :



$$\frac{X_1(s)}{X_2(s)+a\left[U(s)-X_1(s)\right]}=\frac{1}{s}$$

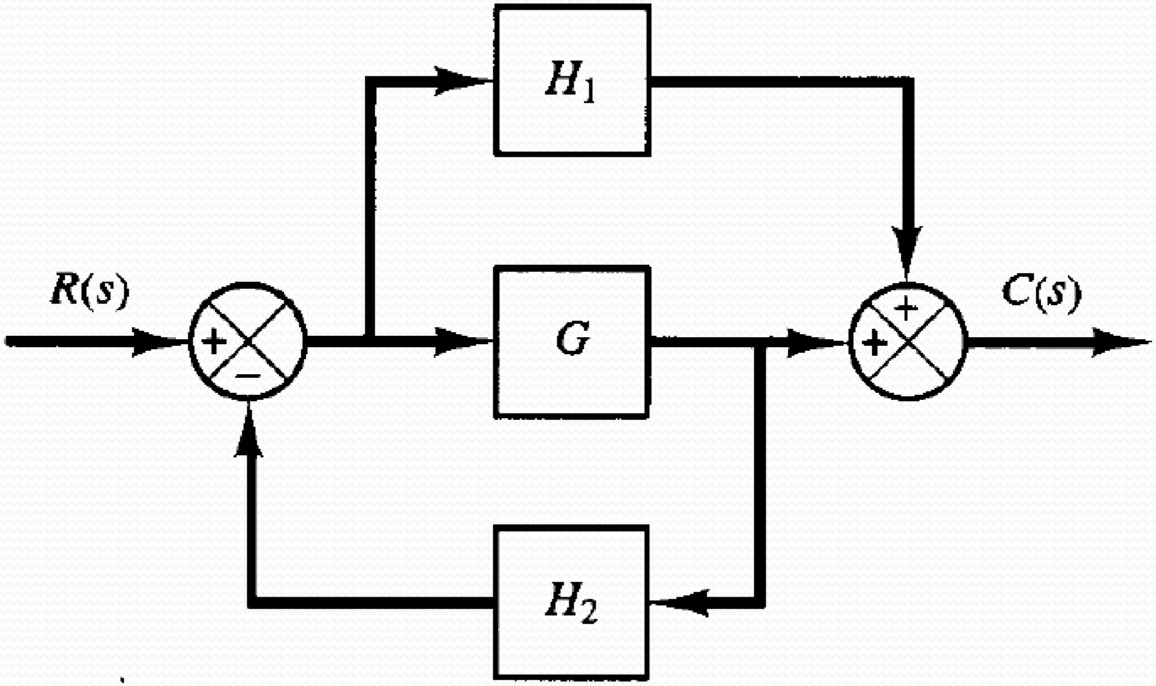
$$\frac{X_2(s)}{U(s)-X_1(s)}=\frac{b}{s}$$

$$Y(s)=X_1(s)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-a & 1\\ -b & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}a\\ b\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Latihan Pra-UTS

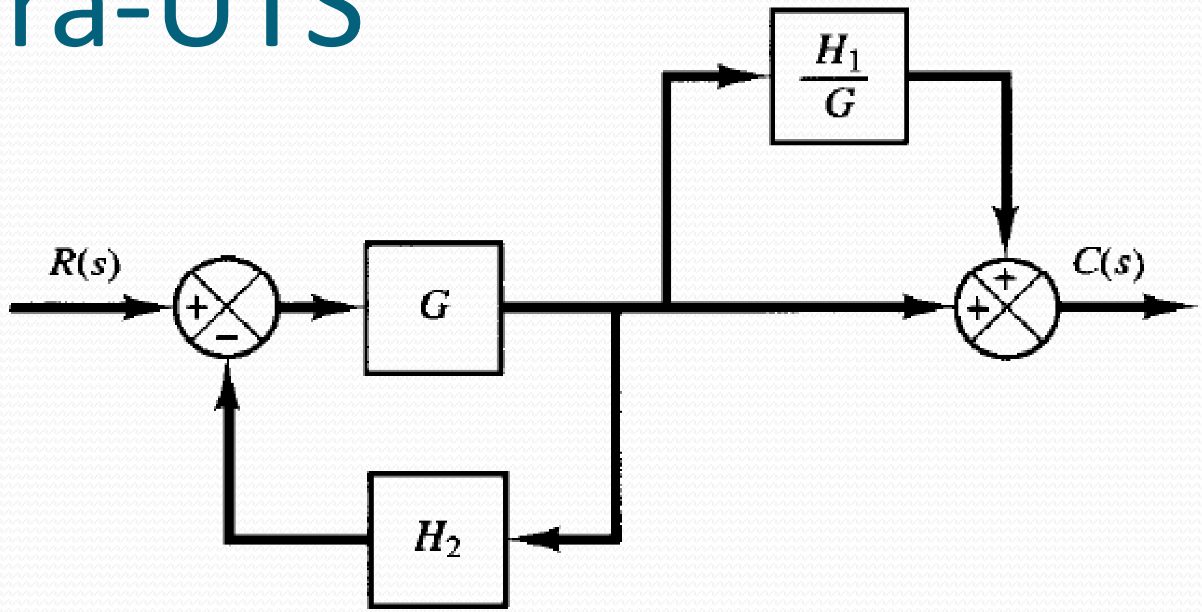


- Simplify the block diagram!

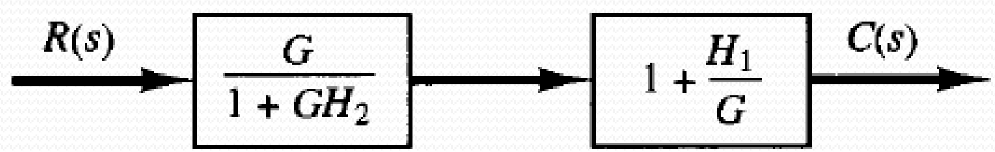
Latihan Pra-UTS

- Answer :

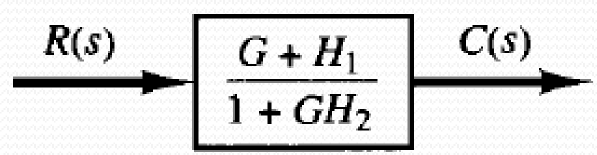
(a)



(b)



(c)



Latihan Pra-UTS

- Obtain the transfer function $G(s)$ of the system defined by the following state-space equation :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Latihan Pra-UTS

Answer :

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{s^2 + 5s + 7} \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Exercise

Answer :

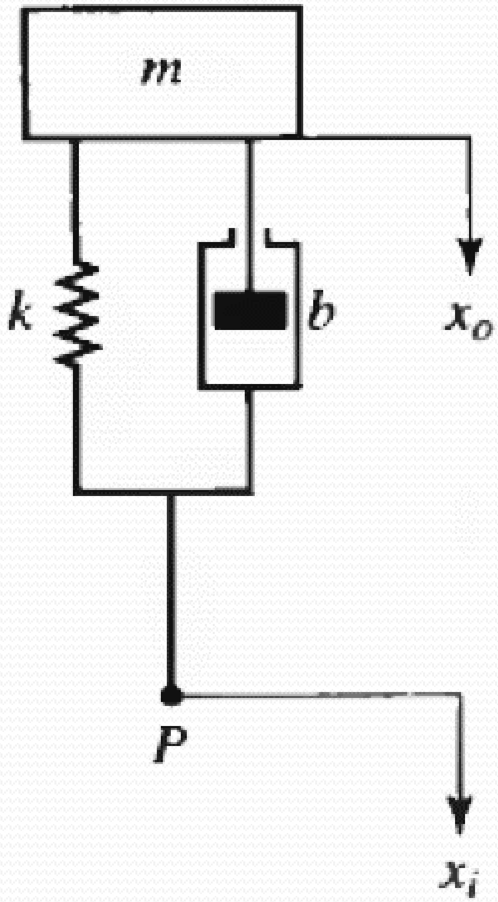
$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{s+1}{s^2+5s+7} & \dfrac{-1}{s^2+5s+7} \\ \dfrac{3}{s^2+5s+7} & \dfrac{s+4}{s^2+5s+7} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} \dfrac{s+1}{s^2+5s+7} & \dfrac{-1}{s^2+5s+7} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$G(s) = \frac{s}{s^2+5s+7}$$

Latihan Pra-UTS

- Obtain transfer function $X_o(s)/X_i(s)$ of the system!



Latihan Pra-UTS

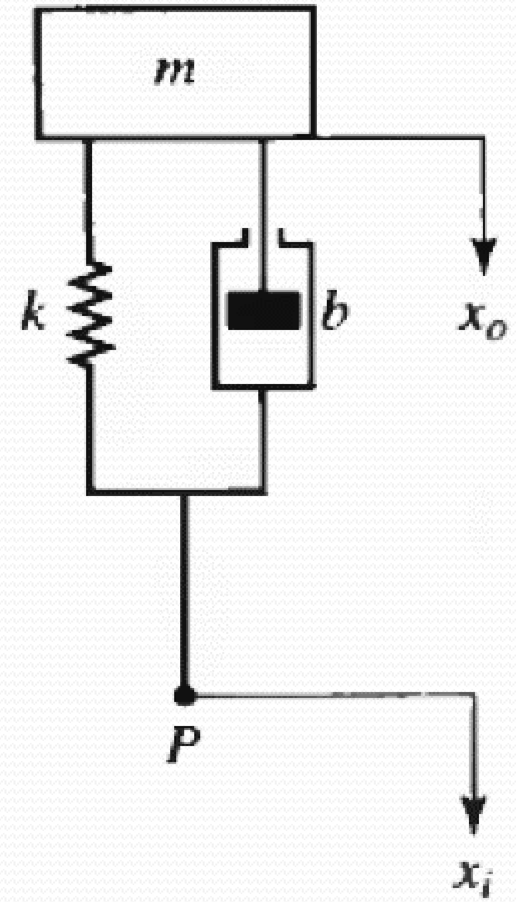
Answer :

The equation for the system is

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$$

or

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i$$



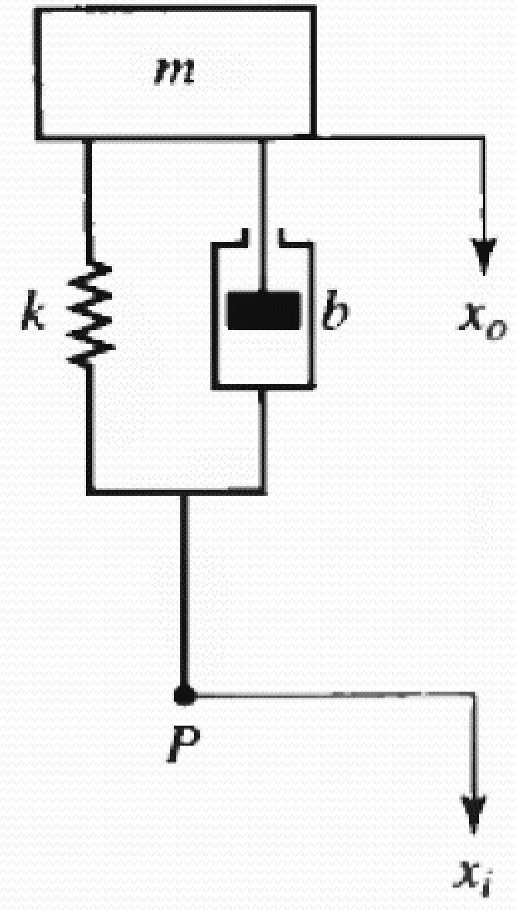
Latihan Pra-UTS

Taking Laplace transform of this last equation, assuming zero initial condition, we obtain

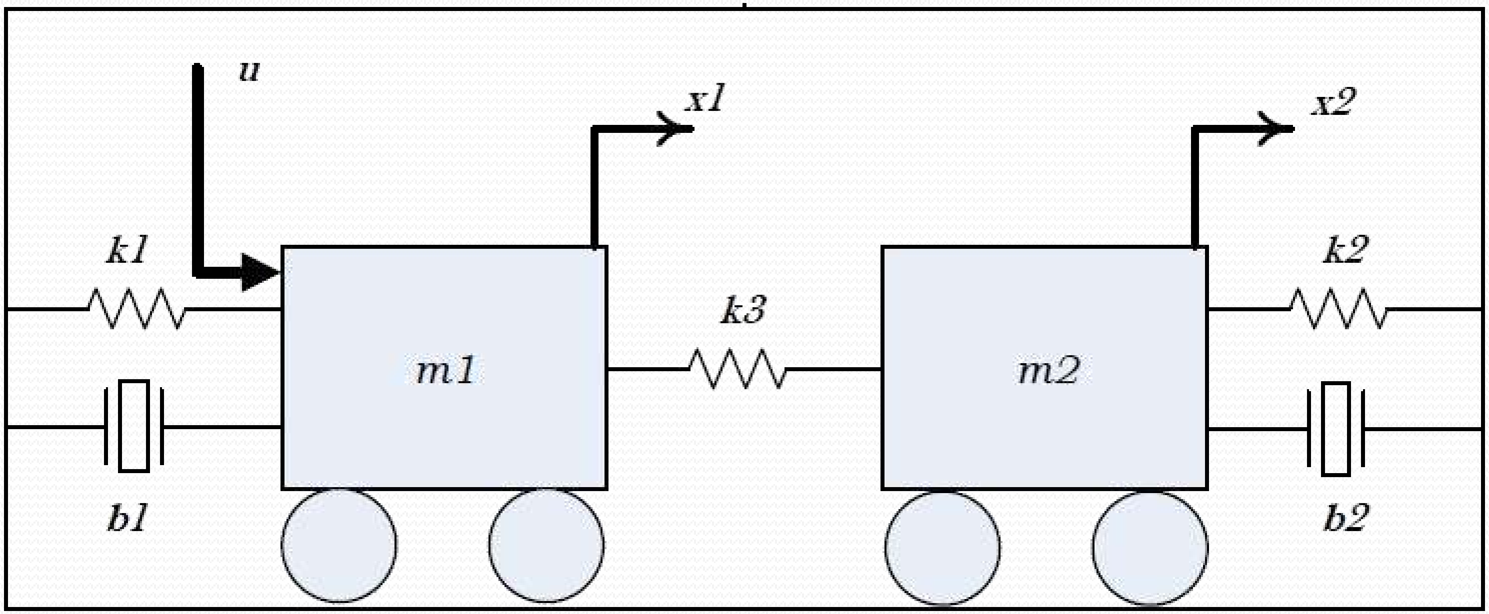
$$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ is given by

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

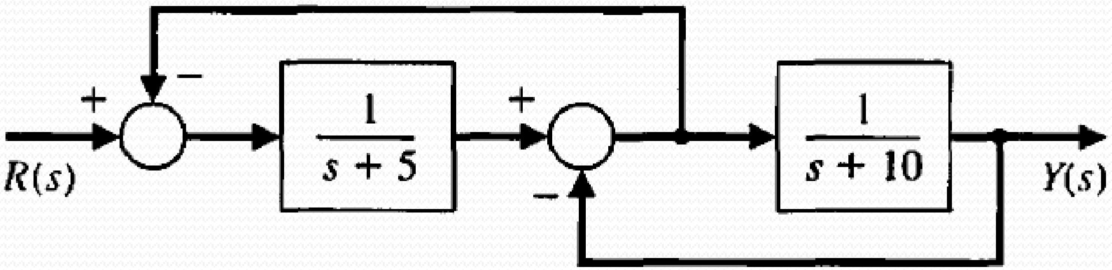


Latihan Pra-UTS

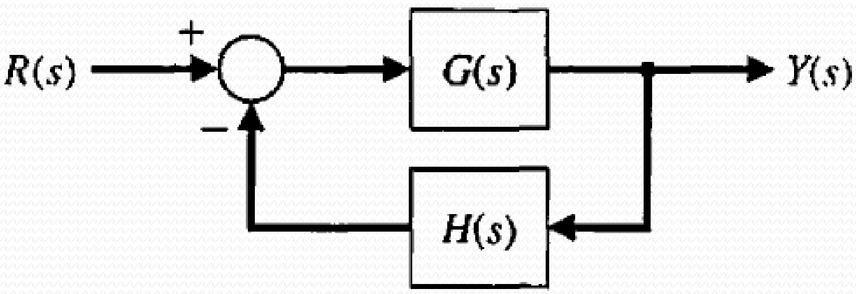


- Obtain transfer function $X_2(s)/U(s)$ of the system!

Latihan Pra-UTS



(a)



(b)

Determine G(s) and H(s) of the block diagram in Fig.(b) that are equivalent to the block diagram in Fig.(a)